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Some exact results for one-dimensional two-band SU(2) bosons

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Abstract. A one-dimensional model for two-band SU(2) bosons with isospin exchange interaction is solved by means of the nested Bethe-ansatz method. The features of the ground state and low-lying excitation state are discussed explicitly by numerical and analytical method. The thermodynamics of the system is analyzed by means of the thermodynamic Bethe-ansatz method, and some physical quantities, such as magnetization, specific heat, etc. are obtained explicitly in some special cases.

PACS. 03.65.-w Quantum mechanics – 72.15.Nj Collective modes (e.g., in one-dimensional conductors) – 03.65.Ge Solutions of wave equations: bound states

1 Introduction

Exactly solvable models play a special role in the condensed matter physics [1], not only because their exact results can be compared with experimental data in an exact way, but also they provide the paradigms that enrich much of our physical intuition. The models typically include Heisenberg chain [2], Hubbard chain [3], and δ-interaction Fermi and Bose gas [4,5] in one dimension. The first one can be solved by means of the algebraic Bethe-ansatz method, while the others can be diagonalized by the nested Bethe-ansatz method. However, in the development of the coordinate Bethe-ansatz method, the problem was only applied to the scalar Bose gas [6], since people generally regard a boson as a particle without internal degree of freedom. While in the recent experiments on Bose-Einstein condensation, a two-component Bose gas has been produced in magnetically trapped ^{87}Rb by rotating the two hyperfine states into each other with the help of slightly detuned Rabi oscillation field [7], and it was noticed that the properties of such Bose system are different from the traditional scalar Bose system once it acquires internal degree of freedom. Thus it is worthwhile to study the problem in the presence of internal degree of freedom. For example, the work [8] co-authored by two of the present authors show that the ground state of two-component bosons in one-dimension differs completely from that of fermion systems, it is a ferromagnetic state for the former, while a singlet for the later.

In this paper we study SU(2)×SU(2) with δ-interaction bosons in one dimension. That is, besides the isospin degree of freedom, each boson possesses two-level band degree of freedom. It is therefore similar to electrons in transition metal oxides [9], which possess spin degree of freedom as well as orbital degree of freedom. In the forthcoming section, we introduce the model, and solve it in term of the nested Bethe-ansatz method. In Section 3, we study the properties of the ground state and show that the ground state of the present model is ferromagnetic both in isospin and band spaces. In Section 4, all possible low-lying excitations are studied both numerically and analytically, four possible elementary excitations are given. The thermodynamics of the system are discussed in Section 5 with the strategy of the thermodynamic Bethe-ansatz [10]. A general formula of the free energy, as well as thermodynamic Bethe-ansatz equations are obtained. In Section 6, some special cases, such as strong coupling limit are studied. Finally, a brief summary and acknowledgement are given in Section 7.

2 The model and its Bethe-ansatz solution

We consider a model of two-band bosons with two-fold internal degree of freedom trapped in one-dimensional ring of length L, whose Hamiltonian reads

$$\mathcal{H} = \int dx \left[\sum_{a,b} \partial_x \psi^*_{a,b} \partial_x \psi_{a,b} + \frac{c}{2} \hat{P}_s \sum_{a,b,a',b'} \psi^*_{a,b} \psi_{a,b} \psi^*_{a',b'} \psi_{a',b'} \right]. \quad (1)$$

Here $a, a' = \uparrow, \downarrow$ and $b, b' = 1, 2$ denote the isospin index and band index respectively, and natural unit is adopted

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for simplicity, c is the coupling constant of isospin exchange $\hat{P}_s$ interaction. At the same time, those fields operators satisfy the bosonic commutation relation with periodic boundary conditions

$$\left[\psi^*_{a,b}, \psi_{a',b'}\right] = \sum_n \delta_{aa'}\delta_{bb'}\delta(x - y - nL). \tag{2}$$

In terms of the group theory, we represent the generators of the isospin $su(2)$ Lie algebra and that of the band $su(2)$ Lie algebra by T and U respectively, i.e. $[T^+, T^-] = 2T^z$ and $[U^+, U^-] = 2U^z$. Then, with the help of these generators, the internal state of the field can be transformed into each other by the following relation

$$T^+|\downarrow\rangle = |\uparrow\rangle, \quad T^-|\downarrow\rangle = |\uparrow\rangle,$$
$$U^+|1\rangle = |2\rangle, \quad U^-|2\rangle = |1\rangle. \tag{3}$$

According to the nested Bethe-ansatz, in domain $x_j \neq x_{j+1}$, we have the Hamiltonian of N free particles, whose eigenstate is simply the superposition of plane waves. Then for a given region with $0 < x_{Q_1} < x_{Q_2} < \cdots < x_{Q_N} < L$, the Bethe-ansatz wave function can be written as:

$$\psi_a(x) = \sum_{P\in S_N} A_{a,b}(P;Q)e^{i(Pk|Qx)}, \tag{4}$$

here $a = (a_1, a_2, \cdots, a_N)$ and $b = (b_1, b_2, \cdots, b_N)$ denote the index-sequence of the isospin and band respectively, $Pk = (k_{P1}, k_{P2}, \cdots, k_{PN})$ is the momentum of all bosons under arbitrary permutation P and $Qx = (x_{Q1}, x_{Q2}, \cdots, x_{QN})$ is the region of all bosons under arbitrary permutation Q, and

$$(Pk|Qx) = k_{P1}x_{Q1} + k_{P2}x_{Q2} + \cdots + k_{PN}x_{QN},$$

For bosonic system, the wave function should be symmetric under permutation of both coordinates and internal degree of freedom, that is

$$(\sigma'_j\psi)_{a,b}(x) = \psi_{a,b}(x). \tag{5}$$

Here $\sigma'\psi$ represents

$$\sigma'_j a : \{a_1 \cdots a_j a_{j+1} \cdots a_N\} \rightarrow \{a_1 \cdots a_{j+1} a_j \cdots a_N\},$$
$$\sigma'_j b : \{b_1 \cdots b_j b_{j+1} \cdots b_N\} \rightarrow \{b_1 \cdots b_{j+1} b_j \cdots b_N\},$$
$$\sigma'_j x : \{x_{Q_1} \cdots x_{Q_j} x_{Q_{j+1}} \cdots x_{Q_N}\} \rightarrow \{x_{Q_1} \cdots x_{Q_{j+1}} x_{Q_j} \cdots x_{Q_N}\}.$$

Using the above relation and the rearrangement theorem in group theory, we have the following consequence from equation (4)

$$A(P;\sigma'Q) = \hat{P}_s\hat{P}_t A(\sigma'P;Q), \tag{6}$$

where $\hat{P}_s$ and $\hat{P}_t$ are the exchange operators for the isospin and band respectively. The discontinuity condition of the first derivative for the wave function at $x_j = x_{j+1}$ plane gives rise to

$$i\big((Pk)_j - (Pk)_{j+1}\big)\Big[A_a(P;\sigma'Q) - A_a(\sigma'P;\sigma'Q) - A_a(P;Q) + A_a(\sigma'P;Q)\Big] = c\hat{P}_s\Big[A_a(P;\sigma'Q) + A_a(\sigma'P;\sigma'Q) + A_a(P;Q) + A_a(\sigma'P;Q)\Big]. \tag{7}$$

Then, together with the relations equations (5) and (7), the scattering matrix of the present model can be written as

$$A(\sigma'P;Q) = \frac{[(Pk)_j - (Pk)_{j+1}] - ic}{[(Pk)_j - (Pk)_{j+1}] + ic}\cdot\frac{[(Pk)_j - (Pk)_{j+1}]\hat{P}_s - ic}{[(Pk)_j - (Pk)_{j+1}] - ic} \times \frac{[(Pk)_j - (Pk)_{j+1}]\hat{P}_t - ic}{[(Pk)_j - (Pk)_{j+1}] - ic}A(P;Q). \tag{8}$$

Applying the periodic boundary conditions $\psi_{a,b}(\ldots, x_{Q_j}, \ldots) = \psi_{a,b}(\ldots, x_{Q_j} + L, \ldots)$ and making use of the standard procedure of the QISM [11], we then obtain the Bethe-ansatz solution of the present model

$$e^{ik_jL} = -\prod_{l=1}^{N}\frac{k_j - k_l + ic}{k_j - k_l - ic}\prod_{\alpha=1}^{M}\frac{k_j - \lambda_\alpha - ic/2}{k_j - \lambda_\alpha + ic/2} \times \prod_{\beta=1}^{M'}\frac{k_j - \nu_\beta - ic/2}{k_j - \nu_\beta + ic/2},$$
$$1 = -\prod_{l=1}^{N}\frac{\lambda_\gamma - k_l + ic/2}{\lambda_\gamma - k_l - ic/2}\prod_{\alpha=1}^{M}\frac{\lambda_\gamma - \lambda_\alpha + ic}{\lambda_\gamma - \lambda_\alpha - ic},$$
$$1 = -\prod_{l=1}^{N}\frac{\nu_c - k_l + ic/2}{\nu_c - k_l - ic/2}\prod_{\beta=1}^{M'}\frac{\nu_c - \nu_\beta + ic}{\nu_c - \nu_\beta - ic}. \tag{9}$$

Here N denotes for the total number of charge rapidities k_j, M for isospin rapidities λ_α, and M' for band rapidities ν_β. Here we would like to point out that the Bethe-ansatz solution just represents the highest (or lowest) weight state in irreducible representation of both isospin SU(2) cross band SU(2), the other states in the subspace can be found by the ladder operators of global SU(2)×SU(2) algebra. And in such a state, there are $N - M$ particles in $|\uparrow\rangle$ and M in $|\downarrow\rangle$; $N - M'$ in $|1\rangle$ and M' in $|2\rangle$.

The logarithm of the equations (9) give rise to

$$k_j L = 2\pi I_j + \sum_{l=1}^{N} \Theta_1(k_j - k_l) + \sum_{\alpha=1}^{M} \Theta_{-1/2}(k_j - \lambda_\alpha) + \sum_{\beta=1}^{M'} \Theta_{-1/2}(k_j - \nu_\beta),$$

$$2\pi J_\gamma = \sum_{l=1}^{N} \Theta_{-1/2}(\lambda_\gamma - k_l) + \sum_{\alpha=1}^{M} \Theta_1(\lambda_\gamma - \lambda_\alpha),$$

$$2\pi J'_c = \sum_{l=1}^{N} \Theta_{-1/2}(\nu_c - k_l) + \sum_{\beta=1}^{M'} \Theta_1(\nu_c - \nu_\beta). \quad (10)$$

Here $\Theta_n(x) = -2\tan^{-1}(x/nc)$. The quantum number I_j of charge rapidities k_j take integer or half-odd-integer depending on $N-M-M'$ is odd or even, while J_γ of isospin rapidities λ_γ and J'_c of band rapidities ν_β take integer or half-odd-integer depending on $N-M$ is odd or even and $N-M'$ is odd or even, respectively. Once the Bethe-ansatz equations are solved, one can calculate the eigenenergy

$$E = \sum_{j=1}^{N} k_i^2, \quad (11)$$

as well as the total momentum

$$P = \frac{2\pi}{L}\left[\sum_{j=1}^{N} I_j - \sum_{\gamma=1}^{M} J_\gamma - \sum_{c=1}^{M'} J'_c\right]. \quad (12)$$

For a large system, it is convenient to define the densities of rapidities $\rho(k), \sigma(\lambda)$ and $\omega(\nu)$,

$$\rho(k_j) = \frac{1}{L(k_{j+1} - k_j)},$$

$$\sigma(\lambda_\gamma) = \frac{1}{L(\lambda_{\gamma+1} - \lambda_\gamma)},$$

$$\omega(\nu_c) = \frac{1}{L(\nu_{c+1} - \nu_c)}. \quad (13)$$

Then the energy and momentum of the system take the form

$$\frac{E}{L} = \int k^2 \rho(k) dk, \quad \frac{P}{L} = \int k\rho(k) dk. \quad (14)$$

and N, M, M' in the equations (9) become

$$\frac{N}{L} = \int \rho(k) dk,$$

$$\frac{M}{L} = \int \sigma(\lambda) d\lambda,$$

$$\frac{M'}{L} = \int \omega(\nu) d\nu. \quad (15)$$

In terms of the group theory, the highest weight state of SU(2) representation can be labelled by $(N-2M, N-2M')$, so the Zeeman term caused by uniform SU(2) external field h is given by

$$H_{zee} = -\frac{hg_t}{2}(N-2M) - \frac{hg_u}{2}(N-2M') = -\frac{(hg_t + hg_u)L}{2}\int \rho(k)dk + hg_t L \times \int \sigma(\lambda)d\lambda + hg_u L \int \omega(\nu)d\nu, \quad (16)$$

where g_t and g_u are Landé g factors for the isospin and band respectively. If we can take $h_1 = -hg_t, h_2 = -hg_u$, then equation (16) can be rewritten as

$$H_{zee} = \frac{(h_1 + h_2)L}{2}\int \rho(k)dk - h_1 L \times \int \sigma(\lambda)d\lambda - h_2 L \int \omega(\nu)d\nu. \quad (17)$$

3 The ground state

From equation (11), we conclude that charge rapidities at the ground state form a "Fermi" sea and its' center point is zero. Moreover, it is easy to show that the quantum number I_j in equation (10) is a monotonically increasing function of k_j, so the quantum number $\{I_j\}$ at the ground state is given by a set of successive integers or half-odd-integers symmetrically arranged around zero, i.e. $I_{j+1} - I_j = 1$. In order to study the properties of J_γ and J'_c, let us consider equation (10) in the weak-coupling limit $c \to 0^+$. Using $\lim_{c\to 0^+} \Theta_{\pm n}(x) \to \mp\pi \text{sgn}(x)$, equation (10) becomes

$$2I_j = \frac{k_j L}{\pi} + \sum_{l=1}^{N} \text{sgn}(k_j - k_l) - \sum_{\alpha=1}^{M} \text{sgn}(k_j - \lambda_\alpha) - \sum_{\beta=1}^{M'} \text{sgn}(k_j - \nu_\beta),$$

$$2J_\gamma = \sum_{l=1}^{N} \text{sgn}(\lambda_\gamma - k_l) - \sum_{\alpha=1}^{M} \text{sgn}(\lambda_\gamma - \lambda_\alpha),$$

$$2J'_c = \sum_{l=1}^{N} \text{sgn}(\nu_c - k_l) - \sum_{\beta=1}^{M'} \text{sgn}(\nu_c - \nu_\beta). \quad (18)$$

We choose the subscripts of the rapidities $k_j, \lambda_\gamma, \nu_c$ in a monotonically increasing order as the same as I_j, J_γ, J'_c, then we get

$$2(I_{j+1} - I_j - 1) = \frac{(k_{j+1} - k_j)L}{\pi} - \sum_{\alpha=1}^{M} \big[\mathrm{sgn}(k_{j+1} - \lambda_\alpha) - \mathrm{sgn}(k_j - \lambda_\alpha)\big] - \sum_{\beta=1}^{M} \big[\mathrm{sgn}(k_{j+1} - \nu_\beta) - \mathrm{sgn}(k_j - \nu_\beta)\big],$$
$$2(J_{\gamma+1} - J_\gamma + 1) = \sum_{l=1}^{N} \left[\mathrm{sgn}(\lambda_{\gamma+1} - k_l) - \mathrm{sgn}(\lambda_\gamma - k_l)\right],$$
$$2(J'_{c+1} - J'_c + 1) = \sum_{l=1}^{N} [\mathrm{sgn}(\nu_{c+1} - k_l) - \mathrm{sgn}(\nu_c - k_l)]. \tag{19}$$

Obviously, for $I_{j+1} - I_j = n$, there will be $k_{j+1} - k_j = 2\pi(n-1)/L$, if there exists a λ_γ or a ν_β between k_j and k_{j+1}, then $k_{j+1} - k_j = 2\pi n/L$. For$J_{\gamma+1} - J_\gamma = m$, there must exist exactly $m+1$ solutions of k_l satisfying $\lambda_\gamma < k_l < \lambda_{\gamma+1}$, and $J'_{c+1} - J'_c = m'$, then there will be $m'+1$ solutions of k_l satisfying $\nu_c < k_l < \nu_{c+1}$. So we can see that an existing λ_γ or ν_β will suppress the density of states in k-space at $k = \lambda_\gamma$ or $k = \nu_\beta$. Solving equation (10) numerically, we get the density of state in k-space. The result in the absence of isospin rapidity is plotted in Figure 1 and that in the presence of one isospin rapidity is plotted in Figure 2. Thus the more the isospin rapidities or band rapidities exist the higher the energy will be. Consequently the ground state of SU(2)×SU(2) interacting bosons is an isospin-band "ferromagnetic" state. Due to the permutation symmetries, we can use the group theory to characterize the ground state ($M = M' = 0$) with a one-row N-column Young tableau, of which the quantum numbers are

$$\{I_j\} := \left\{-\frac{(N-1)}{2}, \cdots, \frac{(N-1)}{2}\right\} \quad \text{and}$$
$$J_\gamma = J'_c = \text{empty}. \tag{20}$$

Correspondingly, the density $\rho_0(k)$ of the ground state can be written as

$$\rho_0(k) = \frac{1}{2\pi} + \int_{-k_F}^{k_F} K_2(k - k')\rho_0(k')dk', \tag{21}$$

where k_F is the quasi-Fermi momentum determined by the conservation of the total particle number

$$\int_{-k_F}^{k_F} \rho(k)dk = \frac{N}{L},$$

and

$$K_n(x) = \frac{1}{\pi}\frac{nc/2}{n^2c^2/4 + x^2}. \tag{22}$$

In terms of density function, the ground state energy then can be calculated by

$$\frac{E_0}{L} = \int_{-k_F}^{k_F} k^2 \rho_0(k)dk. \tag{23}$$

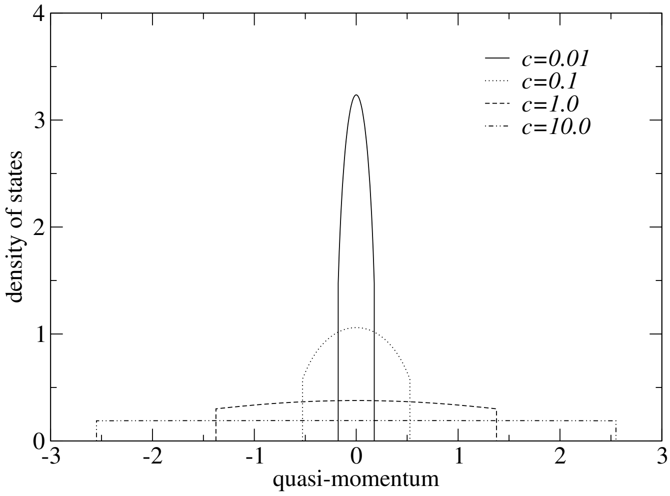

Fig. 1. The density of state in k-space for the ground state. The distribution changes from a histogram to a narrow peak gradually for the coupling from strong to weak. The figure is plotted for $N = L = 100$ and $c = 10, 1, 0.1, 0.01$.

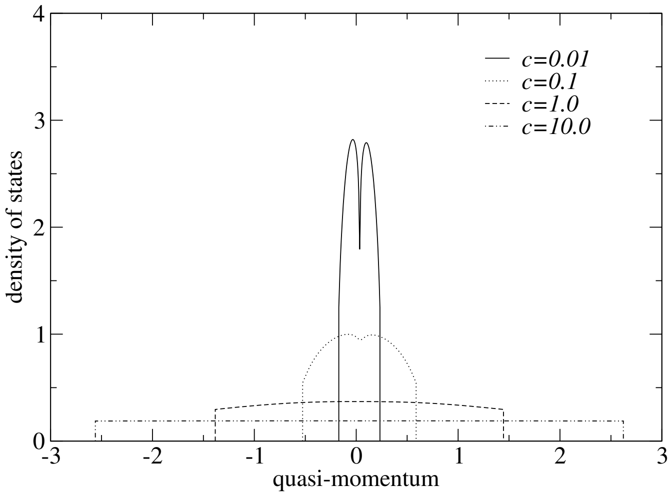

Fig. 2. The density of state in k-space for the ground state in the presence of one isospin rapidity by choosing $J_1 = 1$. The distribution changes from a histogram to a narrow peak gradually for the coupling from strong to weak. The figure is plotted for $N = L = 100$ and $c = 10, 1, 0.1, 0.01$.

It is also useful to consider the problem in strong-coupling limit, i.e. $c \to \infty$. If $M = M' = 0$, then the secular equations become

$$k_j L = 2\pi I_j + \sum_{l=1}^{N} \Theta_1(k_j - k_l), \tag{24}$$

if there exists one λ rapidity (or one ν rapidity), the quantum number of charge rapidities will change from integer to half-integer, or vise versa, that is $I_j - I'_j = 1/2$. Then

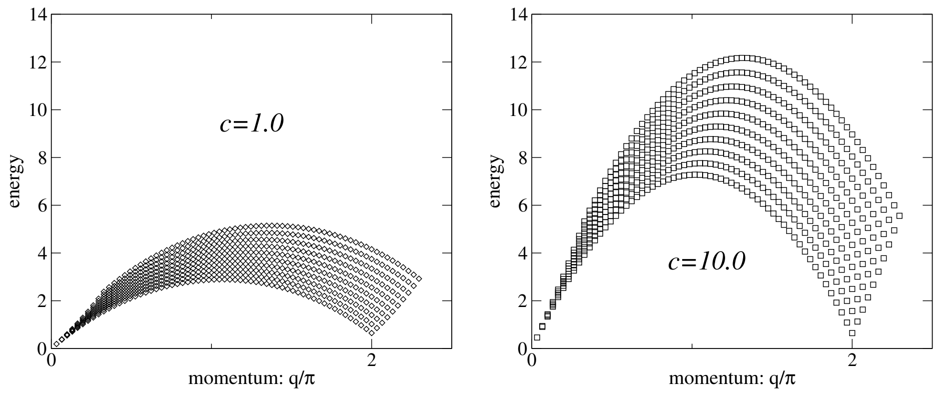

Fig. 3. The holon-particle excitation spectrum calculated for a system of $N = L = 61$.

we have

$$k'_j L = 2\pi I'_j + \sum_{l=1}^{N} \Theta_1(k'_j - k'_l) \quad + \Theta_{-1/2}(k'_j - \lambda_1). \quad (25)$$

As $c \to \infty$, $\tan^{-1} x/c \simeq x/c$, so the above two equations become

$$(k_{j+1} - k_j)L \left[1 + \frac{2N}{Lc}\right] = 2\pi, \quad (26)$$

$$(k'_{j+1} - k'_j)L \left[1 + \frac{2(N-2)}{Lc}\right] = 2\pi.$$

Now we can see that the density distribution $1/L(k_{j+1} - k_j)$ of k for $M = 1$ is smaller than that for $M = 0$. So the eigenenergy of the latter case is larger than the former. This also supports our previous conclusion obtained in weak-coupling limit. Clearly the ferromagnetic ground state of SU(2)×SU(2) Bose system is completely different from that of Fermi system.

4 Low-lying excited states

The low-lying excitation states can be obtained by the variation of the quantum number configuration from that of the ground state.

4.1 Holon-particle excitation

The first case we consider is to remove one I_j from the quantum number configuration of the ground state, then add it outside the sequence. We call it holon-particle excitation, since it creates a holon under the Fermi point and a particle outside it. Obviously, both freedom of isospin and band keep unchanged in this type of excitation, i.e. $M = 0, M' = 0$, and the corresponding quantum number configuration is

$$\{I_j\} = \{-(N-1)/2, \cdots, (N-1)/2, I_n\},$$

where $|I_n| > (N-1)/2$.

We show the numerical result of the momentum-energy spectra of this type of excitation for a system with $L = N = 61$ in Figure 3, from which we can see that there is also a minimum in the spectra at $P = 2\pi$ except $P = 0$. This is because if we replace I_1 or I_N by $(N + 1)/2$, the two excitation states almost have the same energy though their momentum difference is 2π. This phenomena can also be interpreted from the periodic boundary conditions of the system. Moreover, the overall structure of the spectra is not changed very much between $c = 1$ and $c = 10$, this is due to both the excitation energy of holon and particle increase when c increases. Replacing $I_N^0 = (N-1)/2$ by $I_N^0 = (N-1)/2 + n$, $n = 1, 2\cdots$ and keeping the other quantum numbers unchanged, we could get a dispersion relation of particle. While replacing I_n^0 $(n = 1, 2\cdots N)$ in turn by $(N + 1)/2$, we can also obtain the dispersion relation of the holon. In Figure 3, they are just the left boundary and bottom boundary of the energy spectra.

In the thermodynamic limit, if we define $\rho(k) = \rho_0(k) + \rho_1(k)/L$, where $\rho_1(k)/L$ is the variation of the density of the ground state, then the state with one hole inside the quasi-Fermi sea k_F and an additional k_p outside k_F satisfy:

$$\rho_1(k) + \delta(k - \bar{k}) = \int_{-k_F}^{k_F} dk' \rho_1(k') K_2(k - k') + K_2(k - k_p), \quad (27)$$

and the excitation energy becomes $\Delta E = \int k^2 \rho_1(k) dk + k_p^2 = \varepsilon_h(\bar{k}) + \varepsilon_a(k_p)$, where ε_h is holon's energy and $\varepsilon_a(k_p) = \varepsilon_h(k_p)$ is particle's energy, they can be calculated by

$$\varepsilon_h(\bar{k}) = -\bar{k}^2 + \int_{-k_F}^{k_F} k^2 \rho_1^h(k, \bar{k}) dk, \quad (28)$$

$$\rho_1^h(k, \bar{k}) = -K_2(k - \bar{k}) + \int_{-k_F}^{k_F} K_2(k - k') \rho_1^h(k', \bar{k}) dk'. \quad (29)$$

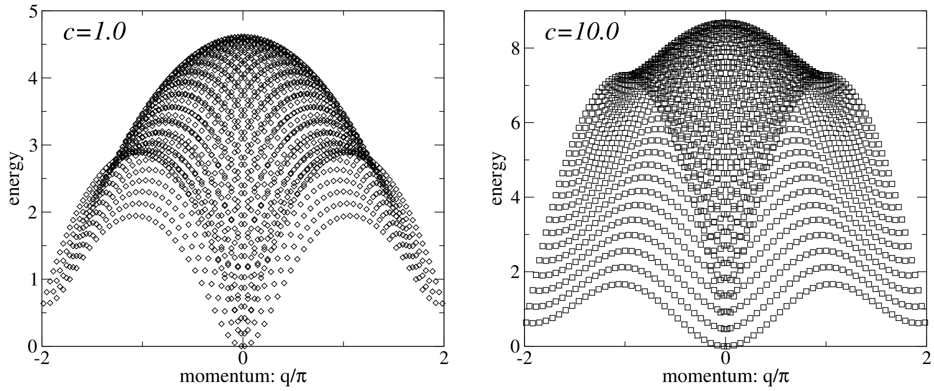

Fig. 4. The holon-isospinon excitation spectrum calculated for a system of $N = L = 61$.

4.2 Holon-isospinon excitation

If we remove one of I_j from the configuration of the ground state and add a λ rapidity to the background of the ferromagnetic ground state, i.e. $(M = 1, M' = 0)$, then we can get the excitation with one hole in the Fermi sea of k rapidities and a particle (we call it isospinon hereafter) in λ space. The corresponding quantum number will be

$$I_1 = -\frac{N}{2} + \delta_{1,j_1}, \quad 1 \le j_1 \le N+1$$
$$I_j = I_{j-1} + 1 + \delta_{j,j_1}, \quad (j = 2, \cdots N). \tag{30}$$

In comparison to the ground state, the quantum number of charge rapidities changes from half-odd-integer to integer, or vice versa. We show the numerical results of momentum-energy spectra of this type of excitation in Figure 4. There exist remarkable difference in the spectra between weak and strong coupling. We interpret this due to that the dependence of the dispersion relation of isospinon and holon on the coupling are quite different.

In the thermodynamic limit, we can use ρ_1 to describe the excitation energy $\Delta E = \int k^2 \rho_1(k) dk$ where

$$\rho_1(k) + \delta(k - \bar{k}) = \int K_2(k - k_l)\rho_1(k_l)dk_l - K_1(k - \lambda_1), \tag{31}$$

the energy of the holon-isospinon excitation can be written down with $\Delta E = \varepsilon_h(\bar{k}) + \varepsilon_c(\lambda)$, ε_h is same to equation (28) and

$$\varepsilon_c(\lambda) = \int k^2 \rho_1^c(k, \lambda) dk, \tag{32}$$

with

$$\rho_1^c(k, \lambda) = -K_1(k - \lambda) + \int_{-k_F}^{k_F} K_2(k - k')\rho_1^c(k', \lambda)dk'. \tag{33}$$

Another type of excitation which is mathematically equivalent to this type of excitation is the holon-bandon excitation, in which an additional ν rapidity is added to the ferromagnetic ground state in stead of λ. Its excitation spectra is the same to that presented in Figure 4. That is to say, a rift emerges at the position of the isospin rapidity for small c that is consistent with our previous analysis for weak coupling.

4.3 Isospinon-isospinon excitation

If we flip the isospin of two bosons down, and keep the ground state configuration of k rapidities unchanged, we can obtain collective excitations with two isospinons in the isospin space. That is $M = 2, M' = 0$ with two additional λ_1, λ_2, whose corresponding quantum number satisfy $-(N-1)/2 < J_1 < J_2 < (N-1)/2$. We show the numerical results of momentum-energy spectra of this type of excitation in Figure 5. Clearly, the whole structure of two spectra in strong-coupling and weak-coupling do not change much. Moreover, differing from the behavior of holon-particle excitation, the excitation energy of isospinon-isospinon decreases when c increases.

In the thermodynamic limit, the excitation energy can be calculated by $\Delta E = \int k^2 \rho_1^c(k, \lambda_1, \lambda_2) dk$, in which $\rho_1^c(k, \lambda_1, \lambda_2)$ satisfy

$$\rho_1^c(k, \lambda_1, \lambda_2) = -K_1(k - \lambda_1) - K_1(k - \lambda_2) + \int_{-k_F}^{k_F} K_2(k - k')\rho_1^c(k', \lambda_1, \lambda_2)dk', \tag{34}$$

4.4 Isospinon-bandon excitation

If we flip one isospin down and one band down, we can obtain another two-parameter excitation, which include one collective mode in isospin space and one in band space, called isospinon and bandon respectively. The energy spectra is determined by $\Delta E = \int k^2 \rho_1^c(k, \lambda_1, \nu_1) dk$ in which $\rho_1^c(k, \lambda_1, \nu_1)$ satisfy

$$\rho_1^c(k, \lambda_1, \nu_1) = -K_1(k - \lambda_1) - K_1(k - \nu_1) + \int_{-k_F}^{k_F} K_2(k - k')\rho_1^c(k', \lambda_1, \nu_1)dk',$$

It is interesting that the numerical results of this type of excitation are almost the same as the isospinon-isospinon excitation (see Fig. 5).

In above, we found that there are four elementary excitations: holon, particle, isospinon, and bandon at low

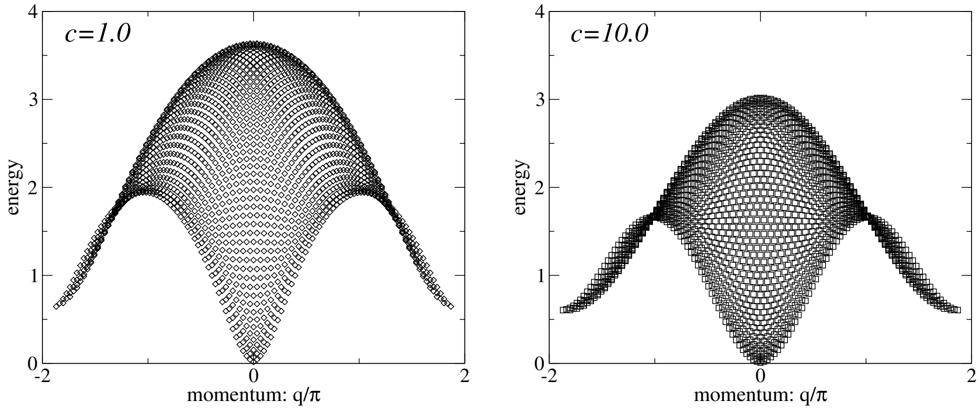

Fig. 5. The isospinon-isospinon excitation spectrum calculated for a system of $N = L = 61$.

energy scale in this system. And from the energy spectra of the above figures, we can conclude that each elementary excitations are all gapless. Moreover, we found that the energy spectra contributed by isospin and band degree of freedom completely differ from that of electronic system with both spin and orbital degree of freedom [12]. This is because, for the later, the singlet ground state form a Fermi sea of quasi particles, which leads to a linear dispersion relation of elementary excitations. While for the former, the collective excitation from the ferromagnetic ground state has a quadratic dispersion relation (see Fig. 5). And in a large system, two elementary excitations in the same sector, therefore, are almost independent. It is just the reason why the spectra of isospinon-isospinon excitation is almost degenerate with that of isospinon-bandon excitation.

5 Thermodynamics at finite temperature

The above discussion are all based on the real roots of the Bethe-ansatz equations, in order to discuss the thermodynamics of the system, we need to consider all possible states described by the Bethe-ansatz solution, among which there may exist complex roots of λ and ν. By considering the consistency of the Bethe-ansatz equations, in the thermodynamic limit, the isospin and band rapidities can form a bound state with another one define by

$$\Lambda_a^{np} = \lambda_a^n + iu(n+1-2p) + O(\exp(-\delta N)),\ p = 1, 2 \cdots n$$
$$\mathrm{U}_a^{np} = \nu_a^n + iu(n+1-2p) + O(\exp(-\delta N)),\ p = 1, 2 \cdots n \quad (35)$$

where $u = c/2$, and they are called λ string and ν string with the length n. λ_a^n and ν_a^n are real parameters representing the motion of the center of mass of the corresponding bound state. Let M_n and M'_n denote the number of λ n-strings and ν n-strings respectively, then the total number of λ and ν are

$$M = \sum_{n=1}^{\infty} n M_n, \quad M' = \sum_{n=1}^{\infty} n M'_n, \quad (36)$$

respectively. Then with the help of equations (35), the Bethe-ansatz equations (10) become

$$\begin{aligned} k_j L &= 2\pi I_j + \sum_{p=1}^{N} \Theta_1(k_j - k_p) \\ &\quad + \sum_{na} \Theta_{-n/2}(k_j - \lambda_a^n) + \sum_{na} \Theta_{-n/2}(k_j - \nu_a^n), \\ 2\pi J_\gamma^n &= \sum_p \Theta_{-n/2}(\lambda_a^n - k_p) + \sum_{pbt\neq 0} \mathrm{A}_{npt} \Theta_{t/2}(\lambda_a^n - \lambda_b^p), \\ 2\pi {J'}_c^n &= \sum_p \Theta_{-n/2}(\nu_a^n - k_p) + \sum_{cpt\neq 0} \mathrm{A}_{npt} \Theta_{t/2}(\nu_a^n - \nu_c^p). \end{aligned} \quad (37)$$

where

$$\mathrm{A}_{npt} = \begin{cases} 1 \text{ for } t = n+p, \mid n-p \mid \\ 2 \text{ for } t = n+p-2, \cdots \mid n-p \mid +2 \\ 0 \text{ otherwise} \end{cases}$$

and quantum number $\{I_j,\ J_\gamma^n,\ {J'}_c^n\}$ label the eigenstate of the system. As a quantum number may be represented as a particle or hole in corresponding space, this determines that the rapidities obey Fermi statistics. In the thermodynamic limit the distribution of rapidities becomes dense and it is useful to introduce density functions for the particles and holes of each class of rapidities. Denoting the densities for charge rapidity by $\rho(k)$ and $\rho^h(k)$, the densities for isospin rapidities by $\sigma_n(\lambda)$ and $\sigma_n^h(\lambda)$, for band rapidities by $\omega_n(\nu)$ and $\omega_n^h(\nu)$, we define:

$$\begin{aligned} \rho(k) + \rho^h(k) &= \frac{1}{L}\frac{dI(k)}{dk}, \\ \sigma_n(\lambda) + \sigma_n^h(\lambda) &= \frac{1}{L}\frac{dJ^n(\lambda)}{d\lambda}, \\ \omega_n(\nu) + \omega_n^h(\nu) &= \frac{1}{L}\frac{d{J'}^n(\nu)}{d\nu}. \end{aligned} \quad (38)$$

Then equations (37) gives rise to the following coupled integral equations

$$\rho + \rho^h = \frac{1}{2\pi} + \int K_2(k-k')\rho(k')dk' - \sum_n \int K_n(k-\lambda)\sigma_n(\lambda)d\lambda - \sum_n \int K_n(k-\nu)\omega_n(\nu)d\nu,$$
$$\sigma_n^h = \int K_2(\lambda-k')\rho(k')dk' - \sum_{pt\neq 0} A_{npt}\int K_t(\lambda-\lambda')\sigma_p(\lambda')d\lambda',$$
$$\omega_n^h = \int K_n(\nu-k')\rho(k')dk' - \sum_{pt\neq 0} A_{npt}\int K_t(\nu-\nu')\omega_p(\nu')d\nu'. \quad (39)$$

In terms of the definition (38), the total number of λ and ν rapidities can be rewritten as

$$\frac{M}{L} = \sum_n n\int \sigma_n(\lambda)d\lambda, \quad \frac{M'}{L} = \sum_n n\int \omega_n(\nu)d\nu, \quad (40)$$

and the z-components of T^z and U^z

$$\frac{T^z}{L} = \frac{1}{2}\int \rho(k)dk - \sum_n n\int \sigma_n(\lambda)d\lambda,$$
$$\frac{U^z}{L} = \frac{1}{2}\int \rho(k)dk - \sum_n n\int \omega_n(\nu)d\nu. \quad (41)$$

The thermal equilibrium is obtained by minimizing the free energy $F = E - E_{zee} - T\mathcal{S} - \mu N$ at finite temperature, where μ is the chemical potential, $\mathcal{S}$ is the entropy of the system, and E_{zee} is the zeeman energy. According to the strategy of Yang, the entropy of the present model can be written as

$$\mathcal{S}/L = \int [(\rho+\rho^h)\ln(\rho+\rho^h) - \rho\ln\rho - \rho^h\ln\rho^h]dk + \sum_n \int [(\sigma_n+\sigma_n^h)\ln(\sigma_n+\sigma_n^h) - \sigma_n\ln\sigma_n - \sigma_n^h\ln\sigma_n^h]d\lambda + \sum_n \int [(\omega_n+\omega_n^h)\ln(\omega_n+\omega_n^h) - \omega_n\ln\omega_n - \omega_n^h\ln\omega_n^h]d\nu. \quad (42)$$

and the Zeeman energy is

$$E_{zee} = h_1 T^z + h_2 U^z. \quad (43)$$

The minimization of the free energy ($\delta F = 0$) yields the following integral equations for the energy potentials:

$$\epsilon(k) = k^2 - \mu - \frac{1}{2}(h_1+h_2) - T\int K_2(k-k')\ln(1+e^{-\epsilon(k')/T})dk' + \sum_n K_n(k-\lambda)\ln(1+e^{-\varphi(\lambda)/T})d\lambda + \sum_n K_n(k-\nu)\ln(1+e^{-\kappa(\nu)/T})d\nu, \quad (44)$$
$$\varphi_n(\lambda) = -h_1 n - T\int K_n(\lambda-k)\ln(1+e^{-\epsilon(k)/T})dk + T\sum_{pt\neq 0} A_{npt}\int K_t(\lambda-\lambda')\ln(1+e^{-\varphi(\lambda')/T})d\lambda',$$
$$\kappa_{\mathbf{n}}(\nu) = -h_2 n - T\int K_n(\nu-k)\ln(1+e^{-\epsilon(k)/T})dk + T\sum_{pt\neq 0} A_{npt}\int K_t(\nu-\nu')\ln(1+e^{-\kappa(\nu')/T})d\nu'.$$

where

$$\frac{\rho^h(k)}{\rho(k)} = \delta(k) = e^{\frac{\epsilon(k)}{T}}, \quad \frac{\sigma_n^h(\lambda)}{\sigma_n(\lambda)} = \eta_n(\lambda) = e^{\frac{\varphi_n(\lambda)}{T}},$$
$$\frac{\omega_n^h(\nu)}{\omega_n(\nu)} = \varDelta_n(\nu) = e^{\frac{\kappa_n(\omega)}{T}}. \quad (45)$$

The Fourier transformation of equation (44) are

$$T\ln\delta = k^2 - \mu - \frac{1}{2}(h_1+h_2) - TK_2(k-k') * \ln(1+\delta^{-1}) + \sum_n TK_n(k-\lambda) * \ln(1+\eta_{n-1}), + \sum_n TK_n(k-\nu) * \ln(1+\varDelta_{n-1}),$$
$$\ln\eta_1 = \frac{1}{4u}\mathrm{sech}\frac{\pi\lambda}{2\mathrm{u}} * \left[\ln(1+\delta^{-1})(1+\eta_2)\right],$$
$$\ln\eta_n = \frac{1}{4u}\mathrm{sech}\frac{\pi\lambda}{2\mathrm{u}} * \left[\ln(1+\eta_{\mathrm{n}+1})(1+\eta_{\mathrm{n}-1})\right],$$
$$\ln\varDelta_1 = \frac{1}{4u}\mathrm{sech}\frac{\pi\lambda}{2\mathrm{u}} * \left[\ln(1+\delta^{-1})(1+\Delta_2)\right],$$
$$\ln\varDelta_n = \frac{1}{4u}\mathrm{sech}\frac{\pi\lambda}{2\mathrm{u}} * \left[\ln(1+\Delta_{\mathrm{n}+1})(1+\Delta_{\mathrm{n}-1})\right]. \quad (46)$$

with the asymptotic conditions

$$\lim_{n\to\infty}\left[\ln\frac{\eta_n}{n}\right] = -\frac{h_1}{T}, \quad \lim_{n\to\infty}\left[\ln\frac{\varDelta_n}{n}\right] = -\frac{h_2}{T}. \quad (47)$$

From the definition of the Helmholz free energy $F = E - T\mathcal{S}$, it is not difficult to obtain

$$F = \mu - \frac{LT}{2\pi}\int \ln\left[1 + e^{-\epsilon(k)/T}\right] dk \tag{48}$$

and the pressure of the system

$$P = -\frac{\partial F}{\partial L} = \frac{T}{2\pi}\int \ln\left[1 + e^{-\epsilon(k)/T}\right] dk, \tag{49}$$

whose expression is the same as other models but equation (44) that $\epsilon(k)$ fulfils is different.

6 Some special limits

According to the statistic mechanics, once the free energy is obtained explicitly, all other physical quantities can also be evaluated in principle. However, since the explicit expression of $\epsilon(k)$ in equations (44) is still difficult to be solved analytically, we can not analyze the physical properties directly. In the following, we will discuss some special cases, such as high temperature limit and strong coupling limit, etc.

6.1 High-temperature limit

When $T \to \infty$, $\rho/\rho^h = \delta^{-1} \to 1$, we consider all functions $\eta_n(\lambda)$ and $\varDelta_n(\nu)$ are independent of their respective parameter. Using $\lim_{u\to 0} \mathrm{sech}(\pi\lambda/2u)/2u = \delta(\lambda)$ and substituting it into equation (46), we have

$$\begin{aligned} &\eta_1^2 = 1 + \eta_2, \quad \eta_n^2 = (1 + \eta_{n+1})(1 + \eta_{n-1}), \\ &\varDelta_1^2 = 1 + \varDelta_2, \quad \varDelta_n^2 = (1 + \varDelta_{n+1})(1 + \varDelta_{n-1}). \end{aligned} \tag{50}$$

Taking Fourier transform to equation (39)

$$\begin{aligned} \sigma_1 + \sigma_1^h &= \frac{1}{4u}\mathrm{sech}\frac{\pi\lambda}{2\mathrm{u}} * (\rho + \sigma_2^{\mathrm{h}}), \\ \sigma_n + \sigma_n^h &= \frac{1}{4u}\mathrm{sech}\frac{\pi\lambda}{2\mathrm{u}} * (\sigma_{\mathrm{n+1}}^{\mathrm{h}} + \sigma_{\mathrm{n-1}}^{\mathrm{h}}), \\ \omega_1 + \omega_1^h &= \frac{1}{4u}\mathrm{sech}\frac{\pi\lambda}{2\mathrm{u}} * (\rho + \omega_2^{\mathrm{h}}), \\ \omega_n + \omega_n^h &= \frac{1}{4u}\mathrm{sech}\frac{\pi\lambda}{2\mathrm{u}} * (\omega_{\mathrm{n+1}}^{\mathrm{h}} + \omega_{\mathrm{n-1}}^{\mathrm{h}}), \end{aligned} \tag{51}$$

and letting $u = 0$ (considering the weak-coupling limit) then we could get the following relation:

$$\begin{aligned} \sum_n n\sigma_n &= \frac{1}{2}\left[\rho - (n_m + 1)\sigma_{n_m} e^{\frac{-n_m h_1}{T}}\right], \\ \sum_n n\omega_n &= \frac{1}{2}\left[\rho - (n'_m + 1)\omega_{n'_m} e^{\frac{-n'_m h_2}{T}}\right], \end{aligned} \tag{52}$$

here n_m and n'_m are the maximal length of λ string and ν string. In the limit of weak field, equation (52) can be transformed into the expressions of n-strings external field (41), then we can get the z-components magnetization of $SU(2)$

$$\begin{aligned} \frac{T^z}{L} &= \int \frac{n_m + 1}{2}\sigma_{n_m}\left[1 - \frac{n_m h_1}{T} + \frac{1}{2}\frac{n_m h_1}{T}^2 + \cdots\right] d\lambda, \\ \frac{U^z}{L} &= \int \frac{n'_m + 1}{2}\omega_{n'_m}\left[1 - \frac{n'_m h_2}{T} + \frac{1}{2}\frac{n'_m h_2}{T}^2 + \cdots\right] d\nu. \end{aligned} \tag{53}$$

The first term in equation (53) arises from self-magnetization, while others are contributed by the external field. It indicates that magnetic properties of the present model at high temperature is dominated by Curies's law, $\chi \propto \frac{1}{T}$.

6.2 The strong-coupling limit

In the strong coupling limit, ($c \to \infty$), from equation (44) we have:

$$\epsilon(k) = k^2 - \mu - \frac{h_1 + h_2}{2}. \tag{54}$$

Then the free energy can be simplified as

$$\frac{F}{L} = \mu D - \frac{2}{3\pi}\mu^{3/2}\left[1 + \frac{\pi^2 k^2 T^2}{8\mu^2}\right], \tag{55}$$

where the external field is omitted. With the help of the density $\rho = \frac{1}{2\pi}\frac{1}{1+e^{(k^2-\mu)/T}}$, and at $T = 0$, the quasi-Fermi surface is just the square root of the chemical potential, $\mu_0 = \pi^2 D^2$. At low temperature, D is determined by

$$D = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1 + e^{(k^2-\mu)/T}} dk, \tag{56}$$

then the chemical potential satisfy the following relation

$$\mu = \pi^2 D^2 \left[1 - \frac{\pi^2 k^2 T^2}{24\mu^2}\right]^{-2}. \tag{57}$$

At low temperature, the term proportional to T^2 is very small, so it is reasonable to replace μ by μ_0 then

$$\mu = \mu_0 \left[1 - \frac{\pi^2 k^2 T^2}{24\mu^2}\right]^{-2}, \tag{58}$$

and the free energy becomes

$$\frac{F}{L} = \mu_0 D\left[1 + \frac{\pi^2}{12}\frac{k^2 T^2}{\mu_0^2}\right] - \frac{2}{3\pi}\mu_0^{\frac{3}{2}}\left[1 + \frac{\pi^2 k^2 T^2}{4\mu_0^2}\right]. \tag{59}$$

Using $S = -\frac{\partial F}{\partial T}$ and $C_v = \frac{\partial S}{\partial T}$, we find the specific heat at low temperature is Fermi-liquid like

$$S = C_v = \frac{T}{6D}. \tag{60}$$

In the strong-coupling regime for the contribution of the charge rapidities and λ, ν strings to $\epsilon(k)$ vanishes, so the above result is same to the one-component case.

7 Summary

In summary, we solved the problem of SU(2)×SU(2) bosons with repulsive δ interaction in one dimension by the nested Bethe-ansatz method. The corresponding Bethe-ansatz equations are clearly different from that of other boson system with different symmetry. On the basis of these equations, we found that the ground state of this system is "ferromagnetic state", that is, both freedoms of isospin and band are frozen at zero temperature. The low-lying excitations are studied both numerically and analytically, though some of them are the same to that of other bosonic systems. The thermodynamic properties have also been discussed by means of the thermodynamic Bethe-ansatz method. We found that the magnetic properties of the system at high temperature satisfy the Curie's law and the specific heat in the strong coupling limit is a linear function of T at low temperature.

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